

CYCLICITY OF NON VANISHING FUNCTIONS IN THE POLYDISC

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ABSTRACT. We use a special version of the Corona Theorem in several variables, valid when all but one of the data functions are smooth, to generalize to the polydisc results obtained by El Fallah, Kellay and Seip about cyclicity of non vanishing bounded holomorphic functions in large enough Banach spaces of analytic functions on the polydisc determined either by weighted sums of powers of Taylor coefficients or by radially weighted integrals of powers of the modulus of the function.

1. INTRODUCTION

The Hardy space can be seen as a space of square integrable functions on the circle with vanishing Fourier coefficients for the negative integers, a space of holomorphic functions on the unit disk, or the space of complex valued series with square summable moduli, and the interaction between those viewpoints has generated a long and rich history of works in harmonic analysis, complex function theory and operator theory.

The present work aims at generalizing one particular aspect of this to several complex variables: the study of cyclicity of some bounded holomorphic functions under the shift operator in large enough Banach spaces.

1.1. Definitions.

Definition 1. Let $\omega : \mathbb{N}^d \rightarrow (0, \infty)$, where $d \in \mathbb{N}^*$, and $p \geq 1$. We define the Banach space

$$X_{\omega,p} := \left\{ f(z) := \sum_{I \in \mathbb{N}^d} a_I z^I : \|f\|_{X_{\omega,p}}^p := \sum_{I \in \mathbb{N}^d} \left(\frac{|a_I|}{\omega(I)} \right)^p < \infty \right\},$$

with the usual multiindex notation, $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, $I = (i_1, \dots, i_d) \in \mathbb{N}^d$, $z^I := z_1^{i_1} \cdots z_d^{i_d}$.

We also write $|I| := i_1 + \cdots + i_d$. We say that ω is *nondecreasing* if for any I, J , $\omega(I + J) \geq \omega(J)$.

We still have to see in what sense f can be understood as a function of $z \in \Omega \subset \mathbb{C}^n$. The following is elementary to prove.

Lemma 2. *If $\log \omega(I) = o(|I|)$, then for any $z \in \mathbb{D}^n$, the map $X_{\omega,p} \ni f \mapsto f(z)$ is well defined and continuous with respect to the norm $\|\cdot\|_{X_{\omega,p}}$. In particular $X_{\omega,p}$ can be seen as a subset of the space $\mathcal{H}(\mathbb{D}^n)$ of holomorphic functions on the polydisc.*

If $\omega(I) = 1$ for any I , then we obtain the Hardy space $H^2(\mathbb{D}^d)$, which can also be described as the set of functions in the Nevanlinna class of the polydisc with boundary values (radial limits a.e.) on the torus $(\partial\mathbb{D})^d$ which are in $L^2(\partial\mathbb{D})^d$, and

$$\|f\|_{H^2}^2 = \sum_{I \in \mathbb{N}^d} |a_I|^2 = \frac{1}{(2\pi)^d} \int_{(\partial\mathbb{D})^d} |f|^2 d\theta_1 \dots d\theta_d.$$

The standard references for Hardy spaces on polydiscs is [8].

Let λ be a probability measure on $[0, 1]^d$, the elements of which are denoted $r := (r_1, \dots, r_d)$. The unit circle is denoted by \mathbb{T} , so that \mathbb{T}^d is the distinguished boundary of \mathbb{D}^d , endowed with its normalized Haar measure denoted by $d\theta$.

Definition 3. *The weighted Bergman space associated to λ is*

$$\mathcal{B} = \mathcal{B}^p(\lambda) := \left\{ f \in \mathcal{H}(\mathbb{D}^d) : \|f\|_p^p := \int_{[0,1]^d} \int_{\mathbb{T}^d} |f(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})|^p d\theta d\lambda(r) < \infty \right\}.$$

Let $H^\infty(\mathbb{D}^d)$ stand for the set of bounded holomorphic functions on the polydisc. The conditions on λ ensure that $H^\infty(\mathbb{D}^d) \subset \mathcal{B}^p(\lambda)$.

Note that the norms of the monomials are given by moments of the measure λ :

$$\|z^I\|_p^p = \int_{[0,1]^d} r^{pI} d\lambda(r),$$

so that $\log(\|z^I\|_p^{-1})$ is a concave function of I . When $p = 2$, $\mathcal{B}^2(\lambda)$ is a Hilbert space and the monomials z^I form an orthogonal system. Notice that in $X_{\omega,2}$, $\|z^I\|_{\omega,2} = \omega(I)^{-1}$, so that

$$\mathcal{B}^2(\lambda) = X_{\omega,2} \text{ with } \omega(I) := \left(\int_{[0,1]^d} r^{2I} d\lambda(r) \right)^{-1/2}.$$

1.2. Main results. Let X be a Banach space as above, defined by power series or as a weighted Bergman space.

Definition 4. We say that a function $f \in X$ is cyclic if for any $g \in X$, there exists a sequence of holomorphic polynomials (P_n) such that $\lim_{n \rightarrow \infty} \|g - P_n f\|_X = 0$.

Note that using the word “cyclic” is a slight abuse of language, since for $d \geq 2$ we are not iterating a single operator, but taking compositions of the multiplication operators by each of the coordinate functions z_1, \dots, z_d . It is, however, a straightforward generalization of the usual notion of cyclicity under the shift operator $f(z) \mapsto zf(z)$.

By Lemma 2 in the case of power series spaces, or by the mean value inequality in the case of Bergman spaces, the point evaluations are continuous, therefore any cyclic f must verify that $f(z) \neq 0$ for any $z \in \mathbb{D}^d$.

For any $k \in \mathbb{N}$, let

$$\frac{1}{\tilde{\omega}(k)} := \sum_{j=1}^d \|z_j^k\|_X = \sum_{j=1}^d \|z^{ke_j}\|_X,$$

where (e_j) stands for the elementary multiindices of \mathbb{N}^d : $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc, so $ke_j = (0, \dots, 0, k, 0, \dots, 0)$, with k in the j -th place.

When $X = X_{\omega,p}$, notice that $d^{-1} \min_{1 \leq j \leq d} \omega(ke_j) \leq \tilde{\omega}(k) \leq \min_{1 \leq j \leq d} \omega(ke_j)$.

Here are two interesting special cases of our results.

Theorem 5. Suppose that $\lim_{k \rightarrow \infty} \tilde{\omega}(k) = \infty$, that $\log \tilde{\omega}(k) = o(k)$, and that

$$(1) \quad \sum_{k \geq 1} \left(\frac{\log \tilde{\omega}(k)}{k} \right)^2 = \infty.$$

Let $U \in H^\infty(\mathbb{D}^d)$, verifying $U(z) \neq 0$ for any $z \in \mathbb{D}^d$.

- (i) If $d \geq 1$, $p \geq 1$ and $X = \mathcal{B}^p(\lambda)$, then U is cyclic in X .
- (ii) If $d = 2$, $X = X_{\omega,2}$ and ω is nondecreasing, then U is cyclic in X .

When we demand a growth condition of a slightly stronger nature on $\tilde{\omega}$, we can expand the range of spaces where the result applies.

Theorem 6. Let $p \geq 2$. If $\lim_{k \rightarrow \infty} \tilde{\omega}(k) = \infty$, and $\log \tilde{\omega}(k) = o(k)$, and there exists an increasing sequence $n_k \rightarrow \infty$ such that

$$(2) \quad \sup_k \frac{\log \tilde{\omega}(k)}{\sqrt{k}} = \infty,$$

then any zero-free $U \in H^\infty(\mathbb{D}^d)$ is cyclic in $X_{\omega,p}$.

1.3. Previous results. Many results have been proved for the case $d = 1$, and even more for $p = 2$. In one dimension, $\omega = \tilde{\omega}$ of course. When furthermore $p = 2$, $X_{\omega,2}$ has a norm equivalent to that of a Bergman space if and only if $\log \omega(n)$ is a concave function of n [3, Theorem A.2 and Proposition 4.1].

In his seminal monograph [7], N. K. Nikolski proved that if ω is non-decreasing, $\lim_{k \rightarrow \infty} \omega(k) = \infty$, $\log \omega(k) = o(k)$, $\log \omega(n)$ is a concave function of n and

$$(3) \quad \sum_{k \geq 1} \frac{\log \tilde{\omega}(k)}{k^{3/2}} = \infty,$$

then any zero-free $f \in H^\infty(\mathbb{D})$ is cyclic in $X_{\omega,2}$.

Our main inspiration comes from [5], where O. El Fallah, K. Kellay and K. Seip show, still for $d = 1$ and $p = 2$, that (1), with no condition of concavity, is enough to imply cyclicity of any nonvanishing bounded function. Even though (1) is a stronger condition than (3), the concavity condition means that there exist weights to which the new result applies while Nikolski's cannot [5, Remark 2].

The novelty in the present work is of course that we have several variables, and exponents $p \neq 2$. We also notice that it is not necessary to make use of the inner-outer factorization: the much easier Harnack inequality suffices.

1.4. A Corona-like Theorem. As in [5], our main tool is a version of the Corona Theorem. In full generality, this is still a vexingly open question in several variables, be it in the ball or the polydisc. However, following an earlier result of Cegrell [4], a simpler proof [1] gives a Corona-type result in the special case where most of the given generating functions are smooth. That result is enough to yield the required estimates in this instance. For Ω a bounded domain in \mathbb{C}^d , let $A^1(\Omega) := \mathcal{H}(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$. As usual, $H^\infty(\Omega)$ stands for the space of bounded holomorphic functions on Ω , and $\|g\|_\infty := \sup_{z \in \Omega} |g(z)|$.

Theorem 7. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^d , such that the equation $\bar{\partial}u = \omega \in L_{(0,q)}^\infty$, $1 \leq q \leq n$, admits a solution $u = S_q \omega \in L_{(0,q-1)}^\infty(\Omega)$ when $\bar{\partial}\omega = 0$, $\omega \in L_{(0,q)}^\infty(\Omega)$, with the bounds :*

$$\|u\|_\infty \leq E_q \|\omega\|_\infty.$$

There exists a constant $C = C(d, \Omega)$ such that if $N \in \mathbb{N}$, $N \geq 2$, and if $f_j \in A^1(\Omega)$, $1 \leq j \leq N-1$, $f_N \in H^\infty(\Omega)$, verify

$$\sup_{z \in \Omega} \max_{1 \leq j \leq N} |f_j(z)| \leq 1, \quad \inf_{z \in \Omega} \sum_{j=1}^N |f_j(z)| \geq \delta > 0,$$

then there exist $g_1, \dots, g_N \in H^\infty(\Omega)$ such that $\sum_{j=1}^N f_j(z)g_j(z) = 1$ and for $1 \leq j \leq d$,

$$\max_{1 \leq j \leq N} \|g_j\|_\infty \leq C(d, \Omega) N^{4d+2} \frac{\max_{1 \leq j \leq N-1} \|\nabla f_j\|_\infty^d}{\delta^{2d+1}}.$$

Note that the polydisc and the ball verify the hypotheses of the theorem.

1.5. Structure of the paper. We gather some preliminary results and a first reduction of the problem in Section 2. Theorem 7 is proved in Section 5, and used in the proofs of the two main theorems. The relatively easy proof of Theorem 6 is given in Section 3. Theorem 5 will follow from a more general and more technical result, Theorem 14, which is stated and proved in Section 4.

2. AUXILIARY RESULTS.

2.1. Multiplier property.

Definition 8. We shall say that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra for X if there exists $C_m > 0$ such that

$$\forall f \in X, \forall g \in H^\infty(\mathbb{D}^d), gf \in X \text{ and } \|gf\|_X \leq C_m \|g\|_\infty \|f\|_X.$$

Notice that, since constants are in X , this implies that $H^\infty(\mathbb{D}^d) \subset X$.

It is immediate that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra for each $\mathcal{B}^p(\lambda)$ with $C_m = 1$. In the case of $X_{\omega,p}$, since $\|z^I\|_\infty = 1$, an obvious necessary condition is that

$$(4) \quad C_m \omega(I + J) \geq \omega(J),$$

but sufficient conditions are not so easy to state in general.

2.2. Some tools. Our first technical tool is a bound from below for the modulus of a zero-free bounded holomorphic function. For $z \in \mathbb{C}^d$, let $|z| := \max_{1 \leq j \leq d} |z_j|$, and for $z \in \mathbb{D}^d$, $z^* := z/|z| \in \partial\mathbb{D}^d$.

Lemma 9. Let U be a zero-free holomorphic function on \mathbb{D}^d such that $\|U\|_\infty \leq 1$, and $z \in \mathbb{D}^d$. Let $c^2 := \ln \frac{1}{|U(0)|}$. Then, for $k \geq 4c^2$,

$$|U(z)| + |z_1|^k + \dots + |z_d|^k \geq e^{-2c\sqrt{k}}.$$

Proof. The conclusion is obvious if $z = 0$. If not, define a holomorphic function on \mathbb{D} by $f_{z^*}(\zeta) := U(\zeta z^*)$. Then

- $\|f_{z^*}\|_\infty \leq 1$;
- $f_{z^*}(0) = U(0)$;
- $\forall \zeta \in \mathbb{D}, f_{z^*}(\zeta) \neq 0$;
- $f_{z^*}(|z|) = U(z)$.

The Harnack inequality applied to the positive harmonic function $\log |f_{z^*}|^{-1}$ shows that

$$|f_{z^*}(\zeta)| \geq \exp \left(-\frac{1+|\zeta|}{1-|\zeta|} \log \frac{1}{|U(0)|} \right) \geq \exp \left(-\frac{2}{1-|\zeta|} \log \frac{1}{|U(0)|} \right).$$

The computation implicit at the beginning of the proof of [5, Lemma 3] shows that $\inf_{\mathbb{D}} |f_{z^*}(\zeta)| + |\zeta|^k \geq e^{-2c\sqrt{k}}$ as soon as $k \geq 4c^2$; applying this to $\zeta = |z|$ we find

$$|U(z)| + |z_1|^k + \cdots + |z_d|^k \geq f_{z^*}(|z|) + |z|^k \geq e^{-2c\sqrt{k}}.$$

□

Lemma 10. *If $X = X_{\omega,p}$ or $\mathcal{B}^p(\lambda)$ from Definitions 1 or 3 respectively, then the space of polynomials $\mathbb{C}[Z] := \mathbb{C}[Z_1, \dots, Z_d]$ is dense in X .*

Proof. By construction, the polynomials are dense in $X_{\omega,p}$.

For $f \in \mathcal{B}^p(\lambda)$, $r = (r_1, \dots, r_d) \in [0, \infty)^d$, let

$$m_{r,p}(f) := \int_{\mathbb{T}^d} |f(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})|^p d\theta$$

denote the mean of $|f|^p$ on the torus $\mathbb{T}(r)$ of multiradius r . Since $|f|^p$ is plurisubharmonic, this is an increasing function with respect to each component of r . In particular, if we set for any $\gamma \in (0, 1)$, $f_\gamma(z) := f(\gamma z)$, $m_{r,p}(f_\gamma) \leq m_{r,p}(f)$ for each r .

We claim that $\lim_{\gamma \rightarrow 1} \|f - f_\gamma\|_{\mathcal{B}^p(\lambda)} = 0$. Indeed, $\|f - f_\gamma\|_{\mathcal{B}^p(\lambda)} = \int_{[0,1]^d} F_\gamma(r) d\lambda(r)$, where

$$F_\gamma(r) := \int_{\mathbb{T}^d} |f(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d}) - f_\gamma(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})|^p d\theta.$$

Since $|f - f_\gamma|^p \leq C_p(|f|^p + |f_\gamma|^p)$,

$$F_\gamma(r) \leq C_p(m_{r,p}(f) + m_{r,p}(f_\gamma)) \leq 2C_p m_{r,p}(f) \in L^1(d\lambda).$$

Since $f_\gamma \rightarrow f$ uniformly on the torus $\mathbb{T}(r)$ for each r , $F_\gamma(r) \rightarrow 0$ for each r , and we can apply Lebesgue's Dominated Convergence theorem.

For each $\gamma \in (0, 1)$, f_γ is holomorphic on a larger polydisc, so can be uniformly approximated by truncating its Taylor series. □

Lemma 11. *For any $p \geq 2$, and any ω satisfying (4), $\|f\|_{\omega,p} \leq C\|f\|_{H^2}$.*

Proof.

$$\|f\|_{\omega,p}^p = \sum_J \frac{|a_J|^p}{\omega(J)^p} \leq \frac{C_m^p}{\omega(0)^p} \sum_J |a_J|^p \leq \frac{C_m^p}{\omega(0)^p} \left(\sum_J |a_J|^2 \right)^{p/2} = \frac{C_m^p}{\omega(0)^p} \|f\|_{H^2}^p.$$

□

2.3. First reduction. We begin by showing that it is enough to obtain a relaxed version of the conclusion.

Lemma 12. *Let $U \in H^\infty(\mathbb{D}^d)$ be a non-vanishing function.*

If either:

- (1) $H^\infty(\mathbb{D}^d)$ is a multiplier algebra for X ,
- (2) or $X = X_{\omega,p}$, $p \geq 2$ and (4) is satisfied,

and if there exists a sequence $(f_n) \subset H^\infty(\mathbb{D}^d)$ such that

$$\lim_{n \rightarrow \infty} \|1 - f_n U\|_X = 0,$$

then U is cyclic in X .

Proof. By Lemma 10, it is enough to show that we can approximate any polynomial P .

In the case of hypothesis (1), we first show that the constant function 1 can be approximated. Let $\varepsilon > 0$. Take $f \in H^\infty(\mathbb{D}^d)$ such that $\|1 - fU\|_X < \varepsilon/2$. By Lemma 10, we can choose $Q \in \mathbb{C}[Z]$ such that $\|f - Q\|_X \leq \frac{1}{C_m \|U\|_\infty} \frac{\varepsilon}{2}$, where C_m is as in Definition 8. Then

$$\|1 - QU\|_X \leq \|1 - fU\|_X + \|U(f - Q)\|_X < \frac{\varepsilon}{2} + C_m \|U\|_\infty \|f - Q\|_X \leq \varepsilon.$$

To finish the proof, let $P \in \mathbb{C}[Z]$. Choose $Q \in \mathbb{C}[Z]$ such that $\|1 - QU\|_X \leq \varepsilon/(C_m \|P\|_\infty)$. Using once again the fact that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra,

$$\|P - PQU\|_X \leq C_m \|P\|_\infty \|1 - QU\|_X \leq \varepsilon.$$

In the case of hypothesis (2), because of (4), we see that for any function $g \in X_{\omega,p}$,

$$\|z^J g\|_{\omega,p}^p \leq C_m^p \|g\|_{\omega,p}^p.$$

Let $P(z) := \sum_{|J| \leq N} a_J z^J$, then

$$\|P(1 - fU)\|_{\omega,p} \leq \sum_{|J| \leq N} |a_J| \|z^J (1 - fU)\|_{\omega,p} \leq C_m \left(\sum_{|J| \leq N} |a_J| \right) \|1 - fU\|_{\omega,p},$$

which can be made arbitrarily small by choosing f .

Now we need to approach PfU by P_1U , where P_1 is another polynomial. Since $Pf \in H^\infty \subset H^2$, we can approximate it in H^2 by truncated sums of its Taylor expansion. Let Q be such a sum. Then by Lemma 11,

$$\|PfU - QU\|_{\omega,p} \leq C \|PfU - QU\|_{H^2} \leq C \|U\|_\infty \|Pf - Q\|_{H^2},$$

and this last can be made arbitrarily small. \square

3. PROOF OF THEOREM 6

Lemma 13. *If $p \geq 2$ and ω verifies (4), then for any $g \in H^\infty$, $J \in \mathbb{N}^d$,*

$$\|z^J g\|_{\omega,p} \leq \frac{C_m}{\omega(J)} \|g\|_\infty.$$

Proof. Let $g(z) = \sum_{K \in \mathbb{N}^d} g_K z^K$. Then, using the same estimate as in the proof of Lemma 11,

$$\begin{aligned} \|z^J g\|_{\omega,p}^p &= \sum_{K \in \mathbb{N}^d} \frac{|g_K|^p}{\omega(J+K)^p} \leq \sup_{K \in \mathbb{N}^d} \frac{1}{\omega(J+K)^p} \sum_{K \in \mathbb{N}^d} |g_K|^p \\ &\leq \frac{C_m^p}{\omega(J)^p} \|g\|_{H^2}^p \leq \frac{C_m^p}{\omega(J)^p} \|g\|_\infty^p. \end{aligned}$$

□

Proof of Theorem 6.

Let $c^2 := -\ln |U(0)|$ and $B > 2c(2d+1)$. By the hypothesis of the theorem, there exists a strictly increasing sequence $(n_k)_{k \geq 1}$ such that for all k , $\log \tilde{\omega}(n_k) \geq B\sqrt{n_k}$.

By Lemma 9 and Theorem 7, we get $g_j \in H^\infty(\mathbb{D}^d)$, for $j = 1, \dots, d+1$, such that

$$(5) \quad g_{d+1}U + g_1 z_1^{n_k} + \dots + g_d z_d^{n_k} = 1,$$

and

$$(6) \quad \forall j = 1, \dots, d+1, \quad \|g_j\|_\infty \leq C(d) n_k^{d+1} e^{2c(2d+1)\sqrt{n_k}}.$$

Set $f_k := g_{d+1}$, we get

$$\|1 - f_k U\|_{\omega,2} \leq \sum_{j=1}^d \|g_j z_j^{n_k}\|_{\omega,2}$$

and using Lemma 13, this gives

$$\|1 - f_k U\|_X \leq \sum_{j=1}^d \|g_j z_j^{n_k}\|_X \leq C_m \sum_{j=1}^d \frac{\|g_j\|_\infty}{\omega(n_k e_j)} \leq C_m \frac{C(d) n_k^{d+1} e^{2c(2d+1)\sqrt{n_k}}}{\tilde{\omega}(n_k)}.$$

By the choice of B , this tends to 0 as $k \rightarrow \infty$. It only remains to apply lemma 12 to conclude. □

4. PROOF OF THEOREM 5

4.1. Main intermediate result.

Theorem 14. *Let X be a Banach space as in Definitions 1 or 3. Suppose that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra for X . Suppose also that $\lim_{k \rightarrow \infty} \tilde{\omega}(k) = \infty$, that $\log \tilde{\omega}(k) = o(k)$, and that condition (1) holds.*

Then any $U \in H^\infty(\mathbb{D}^d)$, verifying $U(z) \neq 0$ for any $z \in \mathbb{D}^d$ is cyclic in X .

Proof. Now we need to distinguish two cases according to the growth of $\omega(k)$.

Case 1: $\sup_k \frac{\log \tilde{\omega}(k)}{\sqrt{k}} = \infty$.

We perform the same proof as in the proof of Theorem 6, using the fact that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra for X , instead of Lemma 13, to get directly $\|z^J g\|_X \leq \frac{C_m}{\omega(J)} \|g\|_\infty$.

Case 2: $\sup_k \frac{\log \tilde{\omega}(k)}{\sqrt{k}} = B < \infty$. To deal with this more delicate case, we shall need the full power of the proof scheme in [5]. Since our Corona-like estimates are slightly different from those in dimension 1, we first need a refined version of [5, Lemma 1].

Lemma 15. *Let $\tilde{\omega}$ be as in Theorem 14. Let $C_0 > 0$. Then there exists a strictly increasing sequence $(n_k)_{k \geq 1}$ such that*

$$\sum_{k \geq 1} \frac{(\log \tilde{\omega}(n_k))^2}{n_k} = \infty,$$

and for all k , $\log \tilde{\omega}(n_{k+1}) \geq 2 \log \tilde{\omega}(n_k)$ and $\log \tilde{\omega}(n_k) \geq C_0 \log n_k$.

The last condition is the only novelty with respect to [5, Lemma 1].

Proof. First notice that there exists an infinite set $E \subset \mathbb{N}^*$ such that for all $n \in E$, $\log \tilde{\omega}(n) \geq C_0 \log n$. Indeed, if not, for n large enough, we would have

$$\log \tilde{\omega}(n) \leq C_0 \log n \leq n^{1/4},$$

and (1) would be violated.

Now let $n_0 = 1$ and if n_j is defined, let

$$\begin{aligned} n'_{j+1} &:= \min \{n > n_j : \log \tilde{\omega}(n) \geq 2 \log \tilde{\omega}(n_j)\}, \\ n_{j+1} &:= \min \{n > n_j, n \in E : \log \tilde{\omega}(n) \geq 2 \log \tilde{\omega}(n_j)\}. \end{aligned}$$

Obviously, $n_j < n'_{j+1} \leq n_{j+1}$. We claim that

$$S := \sum_{j \geq 0} \sum_{k=n'_j}^{n_{j+1}-1} \left(\frac{\log \tilde{\omega}(k)}{k} \right)^2 < \infty.$$

Accepting the claim, the proof finishes as in [5]:

$$\begin{aligned} \sum_{k \geq 1} \left(\frac{\log \tilde{\omega}(k)}{k} \right)^2 &\leq S + \sum_{j \geq 0} \sum_{k=n_j}^{n'_{j+1}-1} \left(\frac{\log \tilde{\omega}(k)}{k} \right)^2 \\ &\leq S + \sum_{j \geq 0} 4 (\log \tilde{\omega}(n_j))^2 \sum_{k=n_j}^{n'_{j+1}-1} \frac{1}{k^2} \leq S + 4 \sum_{j \geq 0} \frac{(\log \tilde{\omega}(n_j))^2}{n_j - 1}, \end{aligned}$$

so the last sum must diverge.

We now prove the claim. If $n'_j \leq k < n_j$, then $n \notin E$, so for j large enough and $n'_j \leq k < n_j$, $\log \tilde{\omega}(k) \leq k^{1/4}$, thus

$$(7) \quad \sum_{k=n'_j}^{n_j-1} \left(\frac{\log \tilde{\omega}(k)}{k} \right)^2 \leq \sum_{k \geq n'_j} \frac{1}{k^{3/2}} \leq \frac{2}{\sqrt{n'_j} - 1}.$$

The definition of n_j implies that $\tilde{\omega}(n_j) \geq C_0 2^j$, and $n'_{j+1} \notin E$ (if it is distinct from n_{j+1}) so

$$C_0 \log n'_{j+1} > \log \tilde{\omega}(n'_{j+1}) \geq 2 \log \tilde{\omega}(n_j) \geq C_0 2^j,$$

and the series with general term the last expression in (7) must converge. \square

We follow the proof of [5, Theorem 1], with a couple of wrinkles.

Choose $A := \max(2, \log C(d))$ where $C(d)$ is the constant in (6). Then for c as in (6), let $C_1^2 := (4(2d+1)Ac)^2 + B^2$. We choose $C_0 \geq C_1/c$, and define the sequence (n_j) as in Lemma 15 above. For any given $j_0 \in \mathbb{N}$, let $\alpha_j^2 := \frac{(\log \tilde{\omega}(n_{j_0+j}))^2}{n_{j_0+j}}$,

$$N := \min \left\{ M : \sum_{j=1}^M \alpha_j^2 \geq (4(2d+1)Ac)^2 \right\},$$

$$\lambda_j := \alpha_j \left(\sum_{i=1}^N \alpha_i^2 \right)^{-1/2}.$$

Notice that for any j , $\alpha_j \leq B$ by the hypothesis of Case 2, and that

$$\sum_{i=1}^N \alpha_i^2 \leq \sum_{i=1}^{N-1} \alpha_i^2 + \alpha_N^2 \leq (4(2d+1)Ac)^2 + B^2 = C_1^2,$$

so that $\lambda_j \geq \alpha_j/C_1$. Clearly, $\lambda_j \leq \alpha_j/(4(2d+1)Ac)$.

We write $U_j := U^{\lambda_j^2}$, so that $U = \prod_{j=1}^N U_j$. As above, choose $f_j := g_{d+1}$ satisfying (5) and (6), but with U_j instead of U and n_{j_0+j} instead

of n_k . The quantity c must then be replaced by $c\lambda_j$. We have

$$\|f_j\|_\infty \leq \exp \left(2c(2d+1)\lambda_j\sqrt{n_{j_0+j}} + (d+1)\log n_{j_0+j} + \log C(d) \right).$$

Notice that

$$c\lambda_j\sqrt{n_{j_0+j}} \geq \frac{c}{C_1} \log \tilde{\omega}(n_{j_0+j}) \geq \frac{cC_0}{C_1} \log n_{j_0+j} \geq \log n_{j_0+j},$$

by our choice of C_0 , so that

$$\|f_j\|_\infty \leq \exp A \left(2c(2d+1)\lambda_j\sqrt{n_{j_0+j}} + 1 \right).$$

We finish as in [5]. Let $f := \prod_{j=1}^N f_j$. Since

$$\begin{aligned} 1 - fU &= 1 - \prod_{j=1}^N f_j U_j = \sum_{k=1}^N (1 - U_k f_k) \prod_{j=1}^{k-1} f_j U_j, \\ \|1 - fU\|_X &\leq C_m \sum_{k=1}^N \|1 - U_k f_k\|_X \prod_{j=1}^{k-1} \|f_j U_j\|_\infty \\ &\leq C_m \sum_{k=1}^N \frac{C(d)n_{j_0+k}^{d+1} e^{2c(2d+1)\sqrt{n_{j_0+k}}} }{\tilde{\omega}(n_{j_0+k})} \prod_{j=1}^{k-1} \|f_j U_j\|_\infty \\ &\leq C_m \sum_{k=1}^N \frac{1}{\tilde{\omega}(n_{j_0+k})} \exp \left(A \sum_{j=1}^k (2c(2d+1)\lambda_j\sqrt{n_{j_0+j}} + 1) \right) \\ &\leq C_m \sum_{k=1}^N \frac{1}{\tilde{\omega}(n_{j_0+k})} \exp \left(Ak + \sum_{j=1}^k \frac{1}{4} \log \tilde{\omega}(n_{j_0+k}) \right) \\ &\leq C_m \sum_{k=1}^N \exp \left(Ak - \frac{1}{2} \log \tilde{\omega}(n_{j_0+k}) \right). \end{aligned}$$

Now choose j_0 such that $\log \tilde{\omega}(n_{j_0}) \geq A$, the sum above has terms with better than geometric decrease, so is bounded by $\tilde{\omega}(n_{j_0+1})^{-1/2}$, which can be made arbitrarily small by choosing j_0 large enough. \square

4.2. Proof of Theorem 5. We now obtain cyclicity results as soon as we can prove that $H^\infty(\mathbb{D}^d)$ is a multiplier algebra on the space X . As remarked after Definition 8, this is always the case when $X = \mathcal{B}^p(\lambda)$. So we obtain Theorem 5 (i).

When $d = 2$, $X = X_{\omega,2}$ and ω is increasing, then the multiplication operators by z_1 and z_2 are commuting contractions on a Hilbert space. Ando's theorem generalizes von Neumann's (but only for two operators, unfortunately) and implies that for any $f \in H^\infty(\mathbb{D}^2)$, $\|fg\|_X \leq \|f\|_\infty \|g\|_X$. So we obtain Theorem 5 (ii).

5. PROOF OF THE CORONA THEOREM WITH SMOOTH DATA

We begin by constructing a partition of unity which exploits the smoothness of the data.

Because of the corona hypothesis, and f_j is continuous up to the boundary of Ω , for $1 \leq j \leq N-1$, we have that $g(z) := \sum_{j=1}^{N-1} |f_j(z)|$ is continuous in Ω , and even Lipschitz with a constant controlled by $\max_{1 \leq j \leq N-1} \|\nabla f_j\|_\infty$.

Set

$$U'_N := \{z \in \bar{\Omega} : g(z) < \frac{N-1}{4N}\delta\} \text{ and } U_N := \{z \in \bar{\Omega} : g(z) < \frac{N-1}{2N}\delta\},$$

and

$$U_j := \{z \in \bar{\Omega} : |f_j| > \frac{\delta}{5N}\} \text{ and } U'_j := \{z \in \bar{\Omega} : |f_j| > \frac{\delta}{4N}\}.$$

Then $U'_j \Subset U_j$, $1 \leq j \leq N$.

Lemma 16. *There exist $C_1 > 0$ and $\chi_j \in \mathcal{C}_c^\infty(U_j)$, $j = 1, \dots, N$, such that for $z \in \Omega$, $0 \leq \chi_j \leq 1$, $\sum_{j=1}^N \chi_j(z) = 1$, and*

$$\left| \frac{\chi_j}{f_j} \right| \leq \frac{C_1 N}{\delta}, \quad \|\nabla \chi_j\|_\infty \leq \frac{C_1 N^2}{\delta} \max_{1 \leq i \leq N-1} \|\nabla f_i\|_\infty, \quad j = 1, \dots, N,$$

$$\max_{1 \leq j \leq N} \sup_{z \in \Omega} \frac{|\nabla \chi_j(z)|}{|f_j(z)|} \leq C_1 \frac{N^3}{\delta^2} \max_{1 \leq i \leq N-1} \|\nabla f_i\|_\infty,$$

where C_1 is an absolute constant.

Proof. We can construct a function $\psi_N \in \mathcal{C}_c^\infty(U_N)$ such that $0 \leq \psi_N \leq 1$ and $\psi_N \equiv 1$ on U'_N , with $\|\nabla \psi_N\|_\infty \leq \frac{C}{\delta}$, for instance by composing $|g|$ with an appropriate smooth one-variable function.

We have

$$\mathcal{O} := \bar{U}'_N \cup \bigcup_{j=1}^{N-1} U'_j \supset \bar{\Omega},$$

because for $z \notin \bigcup_{j=1}^{N-1} U'_j$, then

$$\forall j = 1, \dots, N-1, |f_j(z)| \leq \frac{\delta}{4N} \Rightarrow \sum_{j=1}^{N-1} |f_j(z)| \leq \frac{N-1}{4N}\delta \Rightarrow z \in \bar{U}'_N.$$

Now we construct a partition of unity $\{\chi_j\}_{j=1, \dots, N}$ subordinated to $\{U_j\}$ in the usual way: we take a nonnegative function $\psi_j \in \mathcal{C}_c^\infty(U_j)$ such that $\psi_j \leq 1$ everywhere and $\psi_j \equiv 1$ on U'_j , with $\|\nabla \psi_j\|_\infty \leq C \frac{N}{\delta} \|\nabla f_j\|_\infty$. We set

$$\chi_j := \frac{\psi_j}{\sum_{k=1}^N \psi_k}.$$

Since $\sum_{k=1}^N \psi_k \geq 1$, we have $0 \leq \chi_j \leq 1$, $\chi_j \in \mathcal{C}_c^\infty(U_j)$ and $\chi_1 + \cdots + \chi_N = 1$ on $\bar{\Omega}$ and

$$\|\nabla \chi_j\|_\infty \leq C \frac{N^2}{\delta} \max_{1 \leq i \leq N-1} \|\nabla f_i\|_\infty.$$

This yields a partition of unity such that $\frac{\chi_j}{f_j} \in \mathcal{C}^\infty(\bar{\Omega})$ for $1 \leq j \leq N$ and for $j \leq N-1$, $\left| \frac{\chi_j}{f_j} \right| \leq \frac{5N}{\delta}$, because $\text{supp } \chi_j \subset U_j$, where $|f_j| > \frac{\delta}{5N}$ and $\chi_j \leq 1$.

For $j = N$ on the other hand, we have $\text{supp } \chi_N \subset U_N$ and, by the corona hypothesis,

$$z \in U_N \Rightarrow |f_N(z)| \geq \delta - g(z) = \delta - \frac{N-1}{2N}\delta = \frac{N+1}{2N}\delta$$

hence

$$\left| \frac{\chi_N}{f_N} \right| \leq \frac{2N}{(N+1)\delta} \leq \frac{5N}{\delta}.$$

An analogous reasoning yields the bound on $\frac{|\nabla \chi_j|}{|f_j|}$, $1 \leq j \leq N$. \square

Proof of Theorem 7. We shall now go through the Koszul complex method, introduced in this context by Hörmander [6], to obtain the explicit bounds we need. We follow the notations of [2].

Let $\wedge^k(\mathbb{C}^N)$ be the exterior algebra on \mathbb{C}^N , let e_j , $j = 1, \dots, N$, be the canonical basis of $\wedge^1(\mathbb{C}^N)$, and $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$, $\alpha_j \in \{1, \dots, N\}$, the associated basis of $\wedge^k(\mathbb{C}^N)$.

Let L_r^k be the space of bounded and infinitely differentiable differential forms in Ω of type $(0, r)$ with values in $\wedge^k(\mathbb{C}^N)$. The norm on these spaces is defined to be the maximum of the uniform norms of the coefficients.

We define two linear operators on L_r^k .

$$\forall \omega \in L_r^k, \quad R_f(\omega) := \omega \wedge \sum_{j=1}^N \frac{\chi_j}{f_j} e_j \in L_r^{k+1}.$$

We see that $\|R_f \omega\| \leq C_f \|\omega\|$, with

$$(8) \quad C_f := N \sup_{1 \leq j \leq N, z \in \Omega} \left| \frac{\chi_j(z)}{f_j(z)} \right|.$$

The operator $d_f : L_r^{k+1} \rightarrow L_r^k$ is defined by induction and linearity. For $\omega \in L_r^0$, $d_f \omega = 0$. To define the operator on L_r^1 , set $d_f(e_j) := f_j$ and extend by linearity.

To define d_f on L_r^{k+1} , for $e_\alpha \in \wedge^k(\mathbb{C}^N)$, $1 \leq j \leq d$, set

$$d_f(e_\alpha \wedge e_j) := f_j e_\alpha - d_f(e_\alpha) \wedge e_j \in L_r^k.$$

It follows that $\|d_f\|_{\mathcal{L}(L_r^{k+1}, L_r^k)} \leq C(k) \max_{1 \leq j \leq N} \|f_j\|_\infty$.

It is easily seen by induction that $d_f^2 = 0$, $\bar{\partial}d_f\omega = d_f\bar{\partial}\omega$ and

$$d_f\omega = 0 \Rightarrow d_f(R_f\omega) = \omega,$$

i.e. $\lambda = R_f\omega$ is a solution to the equation $d_f\lambda = \omega$ when the necessary condition $d_f\omega = 0$ is verified.

Together with the operator $\bar{\partial} : L_r^k \longrightarrow L_{r+1}^k$, we have a double complex, whose elementary squares are commutative diagrams.

We now construct by induction, for $0 \leq k \leq N$, forms $\omega_{k,l} \in L_l^k$ and $\alpha_{k,l} \in L_l^{k+1}$, where $l \leq k \leq l+1$.

We start with $\omega_{0,0} = 1$,

$$\omega_{1,0} := R_f(\omega_{0,0}) = \sum_{j=1}^N \frac{\chi_j}{f_j} e_j \in L_0^1.$$

Then, if $\omega_{k,k-1}$ is given, we set $\omega_{k,k} := \bar{\partial}\omega_{k,k-1}$; if $\omega_{k,k}$ is given, we set $\omega_{k+1,k} := R_f\omega_{k,k}$. This construction stops for $k = d$ since there are no $(0, d+1)$ forms on \mathbb{C}^d .

Claim. For any $k \geq 0$, $d_f\omega_{k+1,k} = \omega_{k,k}$.

We prove the claim by induction. It is enough to see that $d_f\omega_{k,k} = 0$. For $k = 0$, this is true by construction. For $k \geq 1$, assume the property holds at rank $k-1$. Then

$$d_f\omega_{k,k} = d_f\bar{\partial}\omega_{k,k-1} = \bar{\partial}d_f\omega_{k,k-1} = \bar{\partial}\omega_{k-1,k-1} = \bar{\partial}^2\omega_{k-1,k-2} = 0.$$

From the construction, we have $\|\omega_{k+1,k}\| \leq C_f\|\omega_{k,k}\|$, with C_f defined in (8). Since

$$\omega_{k,k} = \bar{\partial}(R_f\omega_{k-1,k-1}) = \bar{\partial}\left(\omega_{k-1,k-1} \wedge \sum_{j=1}^N \frac{\chi_j}{f_j} e_j\right) = \omega_{k-1,k-1} \wedge \bar{\partial}\left(\sum_{j=1}^N \frac{\chi_j}{f_j} e_j\right)$$

because $\omega_{k-1,k-1}$ is $\bar{\partial}$ -exact, we find $\|\omega_{k,k}\| \leq D'_f\|\omega_{k-1,k-1}\|$, with

$$D'_f := N \sup_{1 \leq j \leq N, z \in \Omega} \frac{\|\nabla \chi_j(z)\|}{|f_j(z)|}.$$

By an immediate induction, $\|\omega_{k,k}\| \leq (D'_f)^k$, $\|\omega_{k+1,k}\| \leq C_f(D'_f)^k$.

We proceed with the construction of the forms $\alpha_{k,l}$, by descending induction. Set $\alpha_{d+2,d} = \alpha_{d+1,d} = 0$. Since $\bar{\partial}\omega_{d+1,d} = 0$ by degree reasons, there exists $u \in L_{d-1}^{d+1}$ such that $\bar{\partial}u = \omega_{d+1,d}$, and $\|u\| \leq E_d\|\omega_{d+1,d}\|$. We set $\alpha_{d+1,d-1} = u$.

Suppose given $\alpha_{k+1,k} = d_f\alpha_{k+2,k}$, with $\bar{\partial}\omega_{k+1,k} - \bar{\partial}\alpha_{k+1,k} = 0$ (this is trivially verified when $k = d$). Then the hypothesis on Ω implies that

there exists $u \in L_{k-1}^{k+1}$ such that

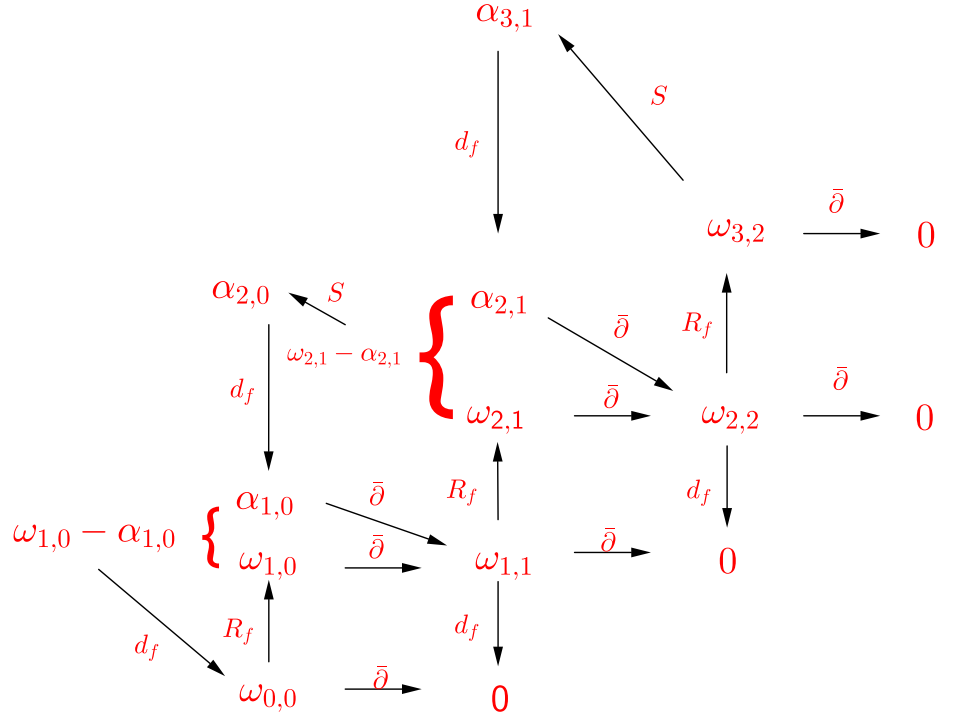
$$\|u\| \leq E_k \|\omega_{k+1,k} - \alpha_{k+1,k}\| \text{ and } \bar{\partial}u = \omega_{k+1,k} - \alpha_{k+1,k}.$$

Then we set $\alpha_{k+1,k-1} := u$.

Finally, we put $\alpha_{k,k-1} := d_f \alpha_{k+1,k-1}$. We need to check the condition on $\bar{\partial}$:

$$\bar{\partial}\alpha_{k,k-1} = d_f \bar{\partial}\alpha_{k+1,k-1} = d_f (\omega_{k+1,k} - d_f \alpha_{k+2,k}) = d_f \omega_{k+1,k} = \omega_{k,k} = \bar{\partial}\omega_{k,k-1}.$$

The following diagram, where \mathcal{S} stands for the operator solving the $\bar{\partial}$ equation, describes the whole complex for $n = 2$, $N = 3$.



The bounds on the solution of the Cauchy-Riemann equation $\bar{\partial}$ and those on $\omega_{k,l}$ imply that

$$\|\alpha_{k+1,k-1}\| \leq E_k (C_f (D'_f)^k + \|d_f\| \|\alpha_{k+2,k}\|),$$

from which we deduce by induction

$$\|\alpha_{k+1,k-1}\| \leq C_f \sum_{j=k}^{d-1} (D'_f)^j \|d_f\|^{j-k} \prod_{i=k}^j E_j + \|d_f\|^d \left(\prod_{j=k}^{d-1} E_j \right) \|\alpha_{d+1,d-1}\|,$$

so taking into account the bound $\|\alpha_{d+1,d-1}\| \leq E_d \|\omega_{d+1,d}\|$, we have for any k

$$\|\alpha_{k+1,k-1}\| \leq C(d) \|f\|_\infty^d \left(\prod_{j=1}^d E_j \right) C_f (D'_f)^d.$$

Finally, we claim that a solution to the Bezout equation is given by the components of $\gamma_{1,0} := \omega_{1,0} - \alpha_{1,0} =: \sum_{j=1}^N g_j e_j$.

Indeed, $\bar{\partial}(\alpha_{1,0} - \omega_{1,0}) = 0$, so the coefficients of $\gamma_{1,0}$ are holomorphic functions, and

$$\sum_{j=1}^N g_j f_j = d_f(\gamma_{1,0}) = d_f(\omega_{1,0} - d_f \alpha_{2,0}) = d_f(\omega_{1,0}) = \omega_{0,0} = 1.$$

The bound on the g_j follows from the bounds on $\|\alpha_{1,0}\|$ and $\|\omega_{1,0}\|$ and Lemma 16, which gives $C_f \leq C \frac{N^2}{\delta}$, $D'_f \leq C \frac{N^4}{\delta^2}$. \square

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